Solution to Assignment 4

Supplementary Exercises

1. Determine which of the following functions are convex/strictly convex:

   (a) $f_1(x) = x^p,\ x \in (0, \infty)$.
   (b) $f_2(x) = x^x\ , x \in (0, \infty)$.
   (c) $f_3(x) = \tan x\ , x \in (-\pi/2, \pi/2)$.
   (d) $f_4(x) = x \log x,\ x \in (0, \infty)$.
   (e) $f_5(x) = (1 + \sqrt{x})^{-1}, \quad x \in (-1, \infty)$.

   Solution. (a) $f_1''(x) = p(p-1)x^{p-2} > 0$ on $(0, \infty)$, so it is strictly convex when $p > 1$ or $p < 0$, convex at $p = 0, 1$, and strictly concave when $p \in (0, 1)$. (A function is concave (resp. strictly concave) if its negative is convex (resp. strictly convex).)

   (b) $f_2''(x) = x^x(1 + \log x)^2 + x^{x-1} > 0$, so it is strictly convex.

   (c) $f_3''(x) = 2\sec^2 x \tan x$ is positive on $(0, \pi/2)$ but negative on $(-\pi, 0)$, so it is not strictly convex on $(-\pi/2, \pi/2)$.

   (d) $f_4''(x) = 1/x > 0$ on $(0, \infty)$, so it is strictly convex.

2. Let f and g be two convex functions defined on I. Show that the function $h(x) = \max\{f(x), g(x)\}$ is convex. Is the function $j(x) = \min\{f(x), g(x)\}$ convex?

   Solution. Let $x, y \in I$ and $\lambda \in n[0, 1]$, we have
   $$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y) \le (1-\lambda)h(x) + \lambda h(y),$$
   and
   $$g((1-\lambda)x + \lambda y) \le (1-\lambda)g(x) + \lambda g(y) \le (1-\lambda)h(x) + \lambda h(y),$$
   and the result follows. The min function is in general not convex. For instance you take $f(x) = x^2$ and $g(x) = (x-1)^2$. Then $j = \min\{f, g\}$ is not convex. Plot the graphs to convince yourself.

3. Give an example to show that the product of two strictly convex functions may not be convex. How about the composite of two strictly convex functions?

   Solution. Take $f(x) = x^{3/2},\ g(x) = x^{-1}$ on $(0, 1)$. Then
   $$f''(x) = \frac{3}{2}\frac{1}{2}x^{-1/2} > 0\ ,\ g'(x) = -x^{-2},\ g''(x) = 2x^{-3} > 0$$
   on $(0, 1)$. Hence $f,\ g$ are convex. Now $(fg)(x) = \sqrt{x}$ on $(0, 1)$ is not convex. (In fact, it is strictly concave as $(fg)''(x) = -x^{-3/2}/2 < 0$.)

   If we consider the composition of two twice differentiable functions $F(G(x))$. We have
   $$\frac{d^2}{dx^2}F(G(x)) = F'(G(x))G''(x) + F''(G(x))G'(x)^2\ .$$
   We see that it is convex provided F and G are both convex and F is increasing. In general, only the convexity of both functions is not sufficient. For instance, consider the function $h(x) = e^{-x^2}$ which is the composition of two strictly convex functions $G(x) = x^2$ and $F(y) = e^{-y}$ but
   $$h''(x) = 2(2x^2 - 1)e^{-x^2}, \quad x \in (-\infty, \infty)\ ,$$
   and is negative, say, at $x = 0$.

4. Let f be a convex function on (a, b) whose inverse exists. Is the inverse function convex?

   No. For instance, $f(x) = x^2$ is strictly convex and strictly increasing on $(0, \infty)$. Its inverse exists and is equal to $g(x) = \sqrt{x}$, $x \in (0, \infty)$. However, g is strictly concave. In general, from the relation

   $$f^{-1}(y) = \frac{1}{f'(x)} \ , \quad y = f(x) \ ,$$

   we see that the slope of the inverse function is decreasing whenever the slope of f is increasing. Therefore, the inverse function of a differentiable convex function with non-vanishing derivative is strictly concave. In general, it can be shown that the inverse of a convex function is concave.